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Extension operators for ultradifferentiable classes of Whitney jets

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In this communication I will talk about Whitney jets. Especially about continuous linear extensions for ultradifferentiable classes of jets defined by intersections. Since all the results are closely related to already known facts of the theory of infinitely differentiable functions, the lead wire will be to draw a parallel between the C^∞ class and those ultradifferentiable classes.

The starting point is Whitney's extension theorem which is the well known "converse Taylor's theorem".

Let us recall that a jet on a compact set K is a sequence of continuous real-valued functions on K. We use this notation :

$$F = \left(F^{(J)}\right)_{J\in\mathbb{N}^n}.$$

For every integer p, for every multi-index J of length $j = |J| \leq p$ and every $\zeta, x \in K$, the Taylor remainder is given by the following expression

$$R_\zeta^{J,p} F(x) = F^{(J)}(x) - \sum_{L\in\mathbb{N};|J+L|\leq p} \frac{1}{L!}(x-\zeta)^L F^{(J+L)}(\zeta).$$

Where $(x-\zeta)^L = (x_1-\zeta_1)^{l_1}...(x_n-\zeta_n)^{l_n}$ and $L! = l_1!...l_n!$.

We say that F is a C^∞ Whitney jet if it satisfies the Taylor's condition :

$$R_\zeta^{J,p} F(x) = o\left(|x-\zeta|^{p-j}\right).$$

A typical C^∞ Whitney jet is given by the restriction of a infinitely differentiable function defined on $\mathbb{R}^n$ by the following linear continuous map :

$$R_K : f \mapsto \left(f_{|K}^{(L)}\right)_{L\in\mathbb{N}^n}.$$

Here the functions $f_{|K}^{(L)}$ are the restrictions of the partial derivatives of f.

The Whitney's extension theorem ensures that the restriction map R_K is surjective. In other words, a Whitney jet comes from a C^∞ function. If K is a singleton, this is the well known Borel's theorem. ([Wh] and [Bo].)

Later, Whitney extension type theorems for ultradifferentiable functions have been proved. An ultradifferentiable class of functions is a set of C^∞ functions with constraints on the growth of the derivatives.

The first result was proved by J. Bruna in [Br] for classes defined by unions (the Carleman classes).

An example of classes defined by intersections is given by the use of weight functions according to the Beurling-Björck approach. In this context, J. Bonet, R. W. Braun, R. Meise and B. A. Taylor have characterized in [$BBMT$] the weight functions for which the analogue of Whitney's theorem holds. With the result of Abanin, the problem was completely solved ([*Ab*]).

Here we consider ultradifferentiable classes built on the model of the intersection of non quasi-analytic Gevrey classes.

Definition. Let ϕ be an increasing convex function on $\mathbb{R}_+$ such that

$$\lim_{t\to+\infty}\frac{\phi(t)}{t}=+\infty \text{ and } \phi(0)=0.$$

i. Let $I_\phi(\mathbb{R}^n)$ denote the class of C^∞-functions f on $\mathbb{R}^n$ such that for every $a>0$,

$$\sup_{P\in\mathbb{N}^n;p=|P|}\ \sup_{x\in\mathbb{R}^n}\frac{\left|f^{(P)}(x)\right|}{p!\exp(\phi(ap))}=\|f\|_a<+\infty.$$

ii. Let $I_\phi(K)$ denote the corresponding space of jets. It's the class of jets F satisfying the following condition :

for every $a>0$,

$$\sup_{P\in\mathbb{N}^n;p=|P|}\ \max\left\{\sup_{x\in K}\frac{\left|F^{(P)}(x)\right|}{p!\exp(\phi(ap))},\ \sup_{\substack{J\in\mathbb{N}^n;\\|J|\le p}}\ \sup_{\substack{(\zeta,x)\in K^2\\ \zeta\neq x}}\frac{\left|R_\zeta^{J,p}F(x)\right|}{j!\exp(\phi(a(p+1)))|\zeta-x|^{p+1-j}}\right\}=\|F\|_a<+\infty.$$

Here, for every multi-index P, $f^{(P)}(x)$ denotes a partial derivative of f.
The topology is given by the norms above. They are Fréchet spaces (projective limit).

Examples.

1. If $\phi(t)=t\ln(1+t)$, it is the intersection of non quasi-analytic Gevrey classes.
2. In fact, the Beurling-Björck approach consistes in replacing $p!\exp(\phi(ap))$ by $\exp\left(\frac{1}{a}\phi_\omega(ap)\right)$ where ω is a weight and ϕ_ω denotes the Young conjugate of the function $t\mapsto\omega(e^t)$. As consequence, for a function ϕ rapidly growing, the Beurling-Björck class coincides with I_ϕ. For instance if $\phi(t)=t^3$.
3. If ϕ is of moderate growth (ie there is $B\in\mathbb{R}_+$ such that for every $t\in\mathbb{R}_+$ we have $\phi(2t)\le Bt+2\phi(t)$) then the classes I_ϕ coincide with the Chaumat-Chollet classes for which good differential analysis properties are satisfied. They prove a Whitney type extension theorem, a Łojasiewicz's regular separation theorem, a Łojasiewicz division theorem and a Glaeser's composition theorem ([*CC*3] and [*CC*4]).

We should mention that the moderate growth implies that their intersections are contained in the intersection of non quasi-analytic Gevrey classes. The classes I_ϕ have been built in order to extend their results. First we have the following theorem which generalizes a theorem of J. Chaumat and A. M. Chollet.

Extension theorem ([*Be*1] and [*Be*2])**.** *Suppose that we have the following complete non quasi-analytic condition :*

$$(H_{cnqa}):\lim_{u\to+\infty}\frac{\phi(u)}{u\ln(\ln u)}=+\infty.$$

Then, for each compact K, the map R_K is surjective from $I_\phi(\mathbb{R}^n)$ onto $I_\phi(K)$, ie, every jet F belonging to $I_\phi(K)$ comes from a function f belonging to $I_\phi(\mathbb{R}^n)$.

Note that the complete non quasi-analytic condition is equivalent to the following condition

$$\forall a > 0, \int_1^{+\infty} \frac{dt}{t \exp\left(\frac{\phi(at)}{t}\right)} < +\infty.$$

Without this condition, by the theorem of Denjoy-Carleman, one cannot choose cutoff function in $I_\phi(\mathbb{R}^n)$.

After, we may ask, under which condition such an extension can be done by a continuous linear operator. Such an operator does not exist in general. Several authors have considered this problem in various situations.

Linear extension.

A. For the C^∞-Whitney jets let us mention some important results :

1. **1. (a)** If K is a singleton there is no continuous linear extension operator.

(b) If $g \in C^\infty(\mathbb{R})$ with $g > 0$ on $\mathbb{R}_+^*$ and $g = 0$ on $\mathbb{R}_-$ we put

$$K_g = \left\{(x, y) \in [0,1]^2, 0 \leq y \leq g(x)\right\}.$$

For this compact set, there is no continuous linear extension operator.
It's a result of M. Tidten $[Ti]$.

2. In the following cases there exists a continuous linear extension operator (with a more or less explicit form).

(a) If K is a segment in $\mathbb{R}$ with non-empty interior.
(B. S. Mityagin, $[Mi]$ 1961).

(b) If K satisfies the following Markov's property :
for every polynomial Q and every multi-index $J \in \mathbb{N}^n$ one has

$$\sup_{x\in K} \left\| Q^{(J)}(x) \right\| \leq \mathcal{M} (\deg Q)^{r|J|} \sup_{x\in K} \|Q(x)\|.$$

with some constants $\mathcal{M}$ and r that do not depend on Q and J.
It's a result of W. Pawłucki et W. Pleśniak $[PP]$ (1988).

(c) A. Goncharov has constructed compact sets of the following form

$$K = \{0\} \cup \left(\bigcup_{k\geq 1} [a_k, b_k] \right)$$

which do not satisfy the Markov's property and at the same time admit a continuous linear extension operator. For instance with this choice of the sequences $a_k = 4 \times 2^{-3^k}$ and $b_k = 6 \times 2^{-3^k}$. ($[Go2]$)

3. A systematic study of the existence of an extension operator in terms of Vogt's invariant was given by Michael Tidten and, more recently, Leonhard Frerick poved that this condition is equivalent to simple interpolative inequalities. ($[Ti]$ and $[Fre]$.)

B. For ultradifferentiable classes of Beurling-Björck type, I recall two important results :

1. By generalizing results of R. Meise and B. A. Taylor, U. Franken gave in $[Fra1]$ a caracterisation of the weight functions for which there is an extension operator for every compact set.

2. For compact sets with Markov's property, U. Franken gave an ultradifferentiable version of the theorem of Pawłucki and Pleśniak. ([$Fra2$])

C. For our intersections I_ϕ, the results are almost similar to the C^∞ case. Here, as in the case of the Beurling-Björck classes, the answer depends on both geometry of the compact K and the growth of the function ϕ.

1. (a) If K is a singleton there is no continuous linear extension operator.

(b) Suppose that ϕ satisfies the complete non quasi-analytic condition

$$\lim_{u\to+\infty}\frac{\phi(u)}{u\ln(\ln u)}=+\infty,$$

then there exists a function $g\in I_\phi(\mathbb{R})$ with $g>0$ on $\mathbb{R}_+^*$ and $g=0$ elsewhere. Put

$$K_g=\left\{(x,y)\in[0,1]^2,0\le y\le g(x)\right\}.$$

For this compact there is no continuous linear extension operator.

2. By adopting a method of Lagrange interpolation polynomials due to W. Pawlucki and W. Plesniak, we get in [$Be1$] the following result.

Theorem. *Let K satisfy Markov's property. If we have*

$$\lim_{u\to+\infty}\frac{\phi(u)}{u\ln u}=+\infty,$$

then there exists a continuous linear extension operator from $I_\phi(K)$ to $I_\phi(\mathbb{R}^n)$.

Remark. For the converse we have only a partial answer. If ϕ is of moderate growth and if K is a segment, then there is no continuous linear extension operator. The moderate growth implies that the quotient $\frac{\phi(u)}{u\ln u}$ is bounded as $u\to+\infty$.

3. For particular compact sets in $\mathbb{R}$, we can use Mityagin's and Goncharov's methods to obtain an explicit form of an extension operator. In each case there are two steps.
- The construction of a basis of the space of jets.
- By extending the elements of the basis we obtain an extension operator.

(a) For $K=[-1,1]$ (similar to the Mityagin's case). We consider the Chebyshev polynomials

$$T_n(x)=\cos(n\arccos x)$$

and we set

$$a_0(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\cos t)\,dt$$

and, for every $n\in\mathbb{N}^*$,

$$a_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\cos t)\cos(nt)\,dt.$$

Result 1 ([$Be3$], [$Be4$])**.** *The system $\left\{(T_n)_{n\ge0},(a_n)_{n\ge0}\right\}$ forms a Schauder basis of the Fréchet space $I_\phi([-1,1])$. ie*

i. $\left\{(T_n)_{n\ge0},(a_n)_{n\ge0}\right\}$ is a biorthogonal system.

ii. for every $f\in I_\phi([-1,1])$, one has

$$f=\sum_{n=0}^{+\infty}a_n(f)\,T_n,$$

iii. for every $b>0$, there exists $a>0$ and $C>0$ such that, for every $f\in I_\phi([-1,1])$, one has $\sum_{n=0}^{+\infty}|a_n(f)|\,\|T_n\|_b\leq C\|f\|_a$.

The proof is not difficult. By the theorem of Dirichlet (with the function $t\mapsto f(\cos t)$), if $f\in I_\phi([-1,1])$ we have

$$\forall x\in[-1,1]\,,f(x)=\sum_{n=0}^{+\infty}a_n(f)\,T_n(x)\,.$$

So we have only to prove the last condition.
If $f\in I_\phi([-1,1])$, for every $n>0$, after r integrations by parts we have

$$a_n(f)=\pm\frac{1}{\pi n^r}\int_{-\pi}^{\pi}(f\circ\cos)^{(r)}(t)\left\{\begin{array}{c}\cos(nt)\\ \text{or}\\ \sin(nt)\end{array}\right\}dt$$

($\cos(nt)$ if r is even; $\sin(nt)$ if r is odd.) By the formula of Faà di Bruno we have

$$\frac{\left|(f\circ\cos)^{(r)}(t)\right|}{r!}=\sum_{i=1}^{r}\left(\frac{f^{(i)}(\cos(t))}{i!}\sum_J C_{J,i,r}\prod_{s=1}^{r}\left(\frac{(\cos)^{(s)}(t)}{s!}\right)^{j_s}\right)$$

with

$$\sum_J C_{J,i,r}=\binom{r-1}{i-1}.$$

The definition of the norm $\|f\|_a$ implies that

$$\left|\frac{f^{(i)}(\cos(t))}{i!}\right|\leq\|f\|_a\exp(\phi(ai))$$

Whereof we get

$$\sup_{t\in\mathbb{R}}\frac{\left|(f\circ\cos)^{(r)}(t)\right|}{r!}\leq\|f\|_a\,2^{r-1}\exp(\phi(ar))\,.$$

Replacing r by $r+2$, we get, for every $a\in\mathbb{R}_+^*$, every $n\in\mathbb{N}^*$ and every $r\in\mathbb{N}$:

$$|a_n(f)|\leq\frac{1}{n^{r+2}}\|f\|_a\,2^{r+2}(r+2)!\exp(\phi(a(r+2)))\,.$$

By using the W. A. Markov's inequalities we get :
for every $n\in\mathbb{N}$ and every $p\in\mathbb{N}$, one has

$$\sup_{x\in[-1,1]}\left|T_n^{(p)}(x)\right|\leq T_n^{(p)}(1)\leq\frac{n^{2p}}{p!}.$$

From this, we prove that there is B such that, for every $n\in\mathbb{N}^*$, we have

$$|a_n(f)|\,\|T_n\|_{8a}\leq B\|f\|_a\frac{1}{n^2}.$$

This implies that

$$\sum_{n=0}^{+\infty}|a_n(f)|\,\|T_n\|_{8a}\leq C\|f\|_a\,.$$

Result 2 $([Be3], [Be4])$**.** *Suppose that* $\lim_{u\to+\infty}\frac{\phi(u)}{u\ln u}=+\infty$*, then, by multiplying each* T_n *whith an appropriate cut-off function* u_n*, we obtain an explicit extension operator* U *by setting* $U(f)=\sum_{n\geq 0}a_n(f)\,u_nT_n$.

Remark. Because a segment satisfies Markov's property this result is not really new. The following theorem, which is the analogue of Goncharov's theorem, shows that in our intersections, as in the C^∞ case, the Markov's property is not necessary.

(b) Let K be a Goncharov compact.

Theorem $([Be3], [Be4])$**.** *Suppose that*

$$\lim_{u\to+\infty}\frac{\phi(u)}{u^2}=+\infty,$$

then the Fréchet space $I_\phi(K)$ *has an absolute basis and there exists a continuous linear extension operator from* $I_\phi(K)$ *to* $I_\phi(\mathbb{R})$*.*

In fact the same functions as in the theorem of Goncharov form a basis in the space $I_\phi(K)$.

Remark. As in the case of ultradifferentiable jets considered for example by M. Valdivia in $[Va]$, in the above mentionned theorems the function can be chosen to be real analytic on the complementary of the compact.

Remark. Recently a generalisation of the extension theorem has been proved by J. Schmets and M. Valdivia (see $[SV]$).

The following table presents the principal results.

Principal results for the C^∞ class and for ultradifferentiable classes defined by intersections[1]

Classes	Extension	Linear extension	Łojasiewicz division theorem
C^∞	If $K = \{0\}$: yes (Borel)	If $K = \{0\}$: no	yes
	For every K : yes (Whitney)	The answer is no, in general If K satisfies Markov's property, then the answer is yes (Pawłucki Pleśniak). Tidten's characterization. Frerick's characterization. See also the Goncharov's example.	
Beurling-Björck	For $K = \{0\}$ or for every K iff the weight is strong (Bonet, Braun, Meise, Taylor, Abanin)	If $K = \{0\}$ explicit necessary and sufficient condition (Meise, Taylor)	The answer is no, in general
		For every K same characterization (Franken)	
		If K satisfies Markov's prop. and for paticular weights (Franken)	
I_ϕ or Chaumat-Chollet (if ϕ is of moderate growth then I_ϕ = Chaumat-Chollet intersections)	If $K = \{0\}$ or for every $K \neq \emptyset$ iff complete non quasi-analytic condition	If $K = \{0\}$: no	yes (Chaumat-Chollet)
		If K satisfies Markov's property and if $\lim_{t\to+\infty} \frac{\phi(t)}{t \ln t} = +\infty$.	
		if K is a compact of Goncharov and if $\lim_{t\to+\infty} \frac{\phi(t)}{t^2} = +\infty$, then the answer is yes.	
Beurling-Komatsu	If $K = \{0\}$: Petzsche's characterization	If $K = \{0\}$: Petzsche's characterization	The answer is no, in general
	if ϕ is of moderate growth then the Beurling-Komatsu and the Beurling-Björck classes coincide We can use the preceding results.		
	Else the question remains open		
Schmets-Valdivia	Generalization of the classes I_ϕ with a Whitney type theorem.		

[1] In this table K denote a compact set.

I'm aware, of course, that the present lecture couldn't possibly be exhaustive; therefore it hasn't been possible for me to quote all the works.

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